Motivation

Relative GW Invariants

Degeneration Formula

Counting Curves in $\mathbb{P}^2$

Recent Developments

Relative Gromov-Witten Invariants and the Degeneration Formula

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Outline

1 Motivation

2 Relative GW Invariants

3 Degeneration Formula

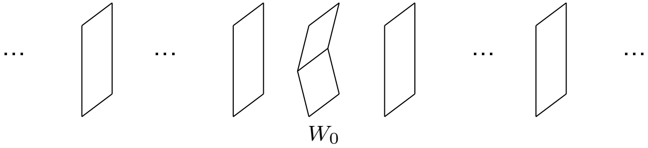
4 Counting Curves in $\mathbb{P}^2$

5 Recent Developments

Motivation

Consider a "nice" family $W \to \mathbb{A}^1$ of varieties:

- Projective family;
- Smooth total space;
- Fibers W_t smooth for $t \neq 0$;
- W_0 is union of two smooth varieties Y_1 and Y_2 that intersect transversally.



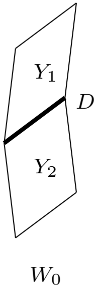
Motivation (cont.)

- Gromov-Witten invariants are deformation-invariant, so we know that the GW theories are equivalent for all W_t with $t \neq 0$.
- So in some sense, the GW theory of W_0 must be equal to that of W_t.
- This correspondence is made precise by Jun Li's theory of relative GW invariants and the degeneration formula.

Motivation (cont.)

Relative GW theory considers the moduli space of relative maps.
Here, "relative" means with respect to a divisor.

The **degeneration formula** computes the GW invariant of W_t for $t \neq 0$ using the relative GW invariants of Y_1 and Y_2 (relative to D).



Relative Maps

- Let Y be a smooth variety and let $D \subset Y$ be a smooth divisor.

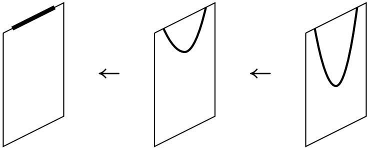
  Write Y^{rel} for the pair (Y, D).

- We say
$$f\colon (X, p_1, \ldots, p_n, q_1, \ldots, q_r) \to Y^{\text{rel}}$$
  is a **relative map** if the p_i's are ordinary marked points and $f(q_j) \in D$ for all j.

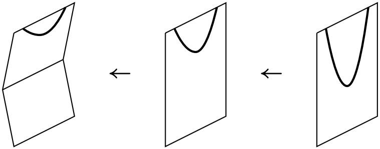
- Fix *contact orders*:
$$f^{-1}(D) = \sum \mu_i q_i$$
  as Cartier divisors for specified set of integers μ_i.

Changing the Target

- The moduli space of relative maps is **not** proper.



- Solution: Define **relative stable morphisms** by changing the target of our morphisms.



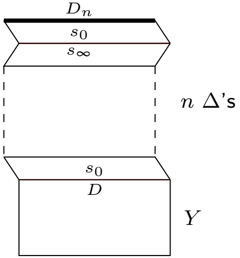
The New Targets

- $\Delta = \mathbb{P}(N_{D/Y} \oplus \mathcal{O}_D)$ is a $\mathbb{P}^1$-bundle over D.
- Let $s_0 = \mathbb{P}(N_{D/Y})$ and $s_\infty = \mathbb{P}(\mathcal{O}_D)$ be the zero and infinity sections of Δ respectively.
- Create Y_n by gluing n copies of Δ to Y.

The New Targets (cont.)

- Glue s_0 of the first copy of Δ to $D \subset Y$.
- Glue s_∞ of the j-th copy of Δ to s_0 of the $j+1$-st copy.
- Let D_n be the infinity section of the last Δ.

Y_n :



Topological Type of Relative Stable Morphisms

Graph: finite collection of vertices, edges, legs, and roots.

- Leg/root: line segment with only one end attached to a vertex.
- Weighted root: a root with a integer assigned to it.

Definition

An *admissible graph* Γ is a graph without edges with the following data:

- Ordering of legs;
- Ordering of weighted roots;
- Two weight functions $g, d: V(\Gamma) \to \mathbb{Z}^{\geq 0}$.

Pre-Stable Relative Morphisms

Let Γ be an admissible graph with l vertices, n legs, and r roots with weights μ_i.

Definition

A *pre-stable relative morphism* to Y of type Γ is a quadruple (f, X, p_i, q_j) such that

1. X is a disjoint union $X_1 \cup \cdots \cup X_l$, where each X_i is a pre-stable connected curve of genus $g(v_i)$.
2. $p_i \in X,\ i = 1, \ldots, n$ and $q_j \in X,\ j = 1, \ldots, r$ are distinct points away from the singular loci of X such that $p_i \in X_j$ (resp. $q_i \in X_j$) if the i-th leg (resp. i-th root) of Γ is attached to v_j.
3. $f\colon (X, p_i, q_j) \to Y_n$ is a relative morphism such that $f^{-1}(D_n) = \sum_{i=1}^{r} \mu_i q_i$ as Cartier divisors and $\deg f|_{X_i} = d(v_i)$.

Admissibility

Let $D_1, \ldots, D_n$ be the singular divisors of Y_n.

Definition

We say that f is *admissible* if the following holds:

1. $f^{-1}(D_i)$ is a finite discrete set;
2. $f^{-1}(D_i)$ is a set of nodes of X;
3. Any such node p is the intersection of two irreducible components, say A_- and A_+, of X such that $f(A_-) \subset \Delta_{i-1}$ and $f(A_+) \subset \Delta_i$.
4. $f|_{A_+}$ and $f|_{A_-}$ have the same contact orders with D_i at $f(p)$.

Relative Stable Morphisms

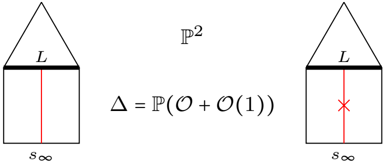
- Define $\mathrm{Aut}(Y_n) = (\mathbb{C}^*)^n$, with $\mathbb{C}^*$ acting on the $\mathbb{P}^1$-fibers of $\Delta_i \subset Y_n$.
- Given a relative map $f\colon X \to Y_n$, define

$$\mathrm{Aut}(f) = \{(h,\sigma) \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y_n) : \sigma \circ f = f \circ h\}.$$

Definition

A pre-stable relative morphism $f\colon X \to Y_n$ is stable if f is admissible and $\mathrm{Aut}(f)$ is finite.

The Stack of Relative Stable Morphisms



$\mathrm{Aut}(f) = \mathbb{C}^*$ $\qquad\qquad$ $\mathrm{Aut}(f) = \{\mathrm{id}\}$

Theorem

The moduli space $\overline{\mathcal{M}}_\Gamma(Y^{rel})$ of relative stable morphisms to $Y^{rel} = (Y, D)$ of fixed topological type Γ is a proper, separated Deligne-Mumford stack.

Relative GW Invariant

Fact: $\overline{\mathcal{M}}_\Gamma(Y^{\mathsf{rel}})$ has a perfect obstruction theory and so admits a virtual fundamental class.

Definition

For $a \in H^*(Y^n)$ and $b \in H^*(D^r)$, the relative GW invariant of topological type Γ is

$$\Psi_\Gamma^{Y^{\mathsf{rel}}}(a,b) = \int_{[\overline{\mathcal{M}}_\Gamma(Y^{\mathsf{rel}})]^{\mathsf{vir}}} \mathrm{ev}_Y^*(a) \cup \mathrm{ev}_D^*(b),$$

where $\mathrm{ev}_Y : \overline{\mathcal{M}}_\Gamma(Y^{\mathsf{rel}}) \to Y^n$ and $\mathrm{ev}_D : \overline{\mathcal{M}}_\Gamma(Y^{\mathsf{rel}}) \to D^r$ are the evaluation maps on the ordinary and relative marked points respectively.

For disconnected domain curves, we take the product of the GW invariants of the connected components.

Returning to the Motivation

- Consider again a projective family $\pi\colon W \to \mathbb{A}^1$ such that:
  - the total space is smooth;
  - the fibers W_t are smooth for $t \neq 0$;
  - W_0 is union of two smooth varieties Y_1 and Y_2 intersecting transversally along smooth divisors $D_i \subset Y_i$.
- Lift cohomology classes $\alpha(t) \in W_t$ to the family:
  - let $\mathbb{Q}_W$ be the sheaf of locally constant functions on W;
  - take $\alpha \in H^0(R^*\pi_*\mathbb{Q}_W)^{\times n}$;
  - let $\alpha(t)$ be its image in $H^*(W_t)^{\times n}$.

Degeneration Formula

Theorem

Let $j_i : Y_i \to W_0$ *be the inclusion map. Then*

$$\Psi^{W_t}_{g,n,d}(\alpha(t)) = \sum_{\gamma} \frac{m(\gamma)}{|\mathrm{Aut}(\gamma)|} \left[\Psi^{Y_1^{rel}}_{\Gamma_1}(j_1^* \alpha(0), b) \cdot \Psi^{Y_2^{rel}}_{\Gamma_2}(j_2^* \alpha(0), b^*) \right],$$

where:

- $\Psi^{W_t}_{g,n,d}$ *is the ordinary GW invariant;*
- $\gamma = (\Gamma_1, \Gamma_2, I)$ *is an admissible triple that glues to give an ordinary stable map to* W_0 *of type* (g, n, d).
- $m(\gamma)$ *is the product of the weights of* γ;
- $\mathrm{Aut}(\gamma)$ *are the automorphisms of the triple; and*
- b^* *is the dual of* b.

The Admissible Triples γ

We say that $\gamma = (\Gamma_1, \Gamma_2, I)$ is an admissible triple if:

1. Each Γ_i corresponds to a relative stable morphism from X_i to Y;
2. Their ordered sets of weighted roots are isomorphic;
   $\Longrightarrow$ This allows us to glue the relative marked points of X_1 and X_2 to produce a new nodal curve X and a new morphism $f: X \to Y$.
3. X is connected of genus g and f is a stable morphism of degree d;
4. I is an ordering of the union of the ordinary marked points of X_1 and X_2 that is consistent with their original orderings.

$m(\gamma)$ and $\mathrm{Aut}(\gamma)$

Recall: $\Psi^W_{g,n,\beta}(\alpha) = \sum_\gamma \frac{m(\gamma)}{|\mathrm{Aut}(\gamma)|}\left(\Psi^{Y_1^{\mathrm{rel}}}_{\Gamma_1}(j_1^*\alpha(0),b)\cdot\Psi^{Y_2^{\mathrm{rel}}}_{\Gamma_2}(j_2^*\alpha(0),b^*)\right).$

Definition

$m(\gamma)$ is the product of the weights of the roots of γ.

Let $\gamma = (\Gamma_1,\Gamma_2,I)$ be an admissible triple with r roots. Then any permutation $\sigma\in S_r$ acts on γ by reordering the roots of γ.

Definition

$\mathrm{Aut}(\gamma) = \{\sigma\in S_r : \gamma = \gamma^\sigma\}.$

Application: Counting Curves in $\mathbb{P}^2$

- Let $\alpha = (\alpha_i)$ and $\beta = (\beta_j)$ be two finite sequences of non-negative integers.
  - $|\alpha| = \sum \alpha_i$.
  - $I\alpha = \sum i\alpha_i$.
  - $I^\alpha = \prod i^{\alpha_i}$.
  - $\alpha + \beta = (\alpha_i + \beta_i)$.
  - $\alpha \geq \beta \iff \alpha_i \geq \beta_i$ for all i.
  - $\binom{\alpha}{\beta} = \prod \binom{\alpha_i}{\beta_i}$.
- Fix a line $L \subset \mathbb{P}^2$.
- Define $N^{d,g}(\alpha, \beta)$ to be the number of degree d, genus g nodal curves that:
  - have contact order i at α_i fixed points of L;
  - have contact order j at β_j arbitrary points of L;
  - pass through an appropriate n general points in $\mathbb{P}^2$.

Caporaso-Harris Formula

- Caporaso & Harris (1998) proved that

$$N^{d,g}(\alpha,\beta) = \sum_{k:\beta_k>0} k \cdot N^{d,g}(\alpha+e_k,\beta-e_k) + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta} \cdot N^{d-1,g'}(\alpha',\beta'),$$

  where the second sum is taken over all α', β', g' such that

  - $\alpha' \leq \alpha$;
  - $\beta' \geq \beta$;
  - $I\alpha' + I\beta' = d-1$;
  - $g - g' = |\beta'| - |\beta| - 1$;
  - $d - 2 \geq g - g'$.

- We will prove this using the degeneration formula. [Ionel & Parker (1998); Li (2004)]

$N^{d,g}(\alpha, \beta)$ as Relative GW Invariants

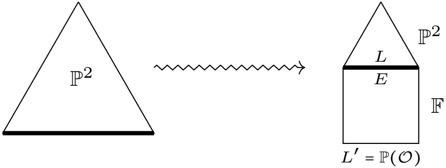
- Let Γ be a graph with:
  - one vertex of degree d and genus g (one connected component);
  - n legs (ordinary marked points);
  - $|\alpha| + |\beta|$ weighted roots (relative marked points).
- Let $a \in H^4((\mathbb{P}^2)^n)$ be the product of point classes and let $b \in H^*(L^{|\alpha|})$ be the product of $|\alpha|$ copies of point classes in $H^2(L)$ and $|\beta|$ copies of $1 \in H^0(L)$. Then

$$N^{d,g}(\alpha, \beta) = \int_{[\overline{\mathcal{M}}_\Gamma(Y,L)]^{\text{vir}}} \text{ev}_{\mathbb{P}^2}^*(a) \cup \text{ev}_L^*(b).$$

- For dimension reasons, the above integral forces $n = 2d + g - 1 + |\alpha| + |\beta|$.

Degeneration

- By blowing up $L \times 0 \subset \mathbb{P}^2 \times \mathbb{A}^1$, we can degenerate $\mathbb{P}^2$ to the union of $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F} = \mathbb{P}(\mathcal{O} \oplus O(1))$, glued along $L \subset \mathbb{P}^2$ and $E \coloneqq \mathbb{P}(\mathcal{O}(1)) \subset \mathbb{F}$.



- More precisely, we are degenerating the relative pair $(\mathbb{P}^2; L)$ to the union of $(\mathbb{P}^2; L)$ and $(\mathbb{F}; E, L')$.
- Obtain the recursive formula of Caporaso-Harris by moving one ordinary marked point p to the $\mathbb{F}$ side.

Outline of Solution

- Let $N_{\mathbb{F}}^{d,g}(\alpha',\beta';\alpha,\beta)$ be the number of curves in $\mathbb{F}$ of degree d and genus g that pass though p and
  - have contact (α',β') along E;
  - have contact (α,β) along L';
- The degeneration formula tells us that

$$N^{d,g}(\alpha,\beta) = \sum_{\gamma/\sim} m(\gamma) N^{d',g'}(\alpha',\beta') \cdot N_{\mathbb{F}}^{d-d',g''}(\beta',\alpha';\alpha,\beta),$$

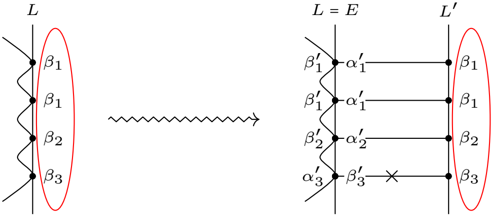
  where $\gamma = (\Gamma_1, \Gamma_2, I)$ is an admissible triple such that:
  - Γ_1 has degree d', genus g', $n-1$ legs, and roots with weights (α',β');
  - Γ_2 has degree $d-d'$, genus g'', 1 leg, and roots with weights (β',α').
- By analyzing classes of curves in $\mathbb{F}$, we can show that $d-d' = 0$ or 1.

$d - d' = 0$

- In $\mathbb{F}$, we must have maps (relative to E and L') of class mF, where F is the fiber class.
- A dimension count forces the map to be fully ramified at E and L, hence the map is unique.
  - If the fiber does not go through the ordinary marked point, the GW invariant is $1/m$.
    In this case, one endpoint must be fixed, and the other must be moving.
  - If the fiber goes through the ordinary marked point, the GW invariant is 1.
    In this case, both endpoints must be moving.

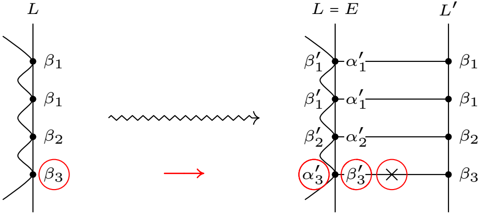
$d - d' = 0$ (cont.)

Example: $(d, g, \alpha, \beta) = (7, 0, (0), (2, 1, 1))$ degenerates to $(7, 0, (0, 1), (2, 0, 1))$.



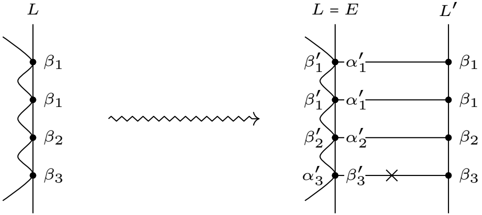
The contact orders on L' are the same as the original contact orders on L.

α' and β'



- Suppose the ordinary marked point is on the last fiber
  $\Longrightarrow$ The contact on E in $\mathbb{F}$ must be moving.
  $\Longrightarrow$ The contact on L in $\mathbb{P}^2$ must be fixed.
- In $\mathbb{P}^2$, a moving contact of order k must change to a fixed contact of order k.
  $\Longrightarrow \alpha' = \alpha + e_k$ and $\beta' = \beta - e_k$.

Relative GW Invariant in $\mathbb{F}$



- Recall:
  - Fiber with no marked point $\Rightarrow$ GW $= 1/m$.
  - Fiber with marked point $\Rightarrow$ GW $= 1$.
- Product of all contact orders $= I^{\alpha'} I^{\beta'} = I^{\alpha} I^{\beta}$.
- $N_{\mathbb{F}}^{0,0}(\beta', \alpha'; \alpha, \beta) = N_{\mathbb{F}}^{0,0}(\beta - e_k, \alpha + e_k; \alpha, \beta) = \frac{k}{I^\alpha I^\beta}$.

$d - d' = 0$ (Plugging In)

So in this case,

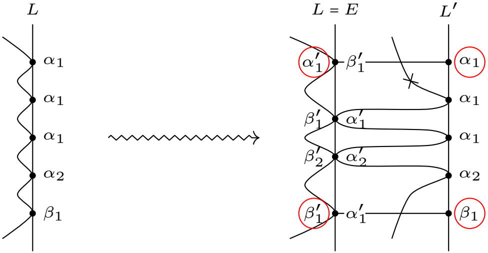
$$
\begin{aligned}
&\sum_{\gamma/\sim} m(\gamma) \cdot N^{d',g'}(\alpha', \beta') \cdot N_{\mathbb{F}}^{d-d',g''}(\beta', \alpha'; \alpha, \beta) \\
&= \sum_{k:\beta_k>0} I^\alpha I^\beta \cdot N^{d,g}(\alpha + e_k, \beta - e_k) \cdot N_{\mathbb{F}}^{0,0}(\beta - e_k, \alpha + e_k; \alpha, \beta) \\
&= \sum_{k:\beta_k>0} I^\alpha I^\beta \cdot N^{d,g}(\alpha + e_k, \beta - e_k) \cdot \frac{k}{I^\alpha I^\beta} \\
&= \sum_{k:\beta_k>0} k \cdot N^{d,g}(\alpha + e_k, \beta - e_k).
\end{aligned}
$$

$d - d' = 1$

- We must have:
  1. Several rational covers of the fiber;
  2. A rational curve in the class $L' + mF$ passing through the ordinary marked point and having all contacts with E and L' fixed.
- Again, there is a unique curve.

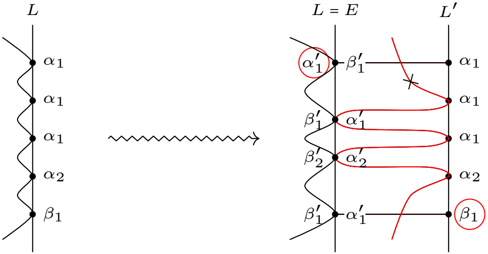
α' and β'

Example: $(d, g, \alpha, \beta) = (6, 0, (3, 1), (1, 0))$ degenerates to $(5, -1, (1.0), (2, 1))$.



- Each β_k on L' must be connected to a β'_k on L by a fiber, so $\beta \leq \beta'$.
- Each α' on L corresponds to a β' on E, which must be on a fiber, so is connected to a α on L'. So $\alpha' \leq \alpha$.

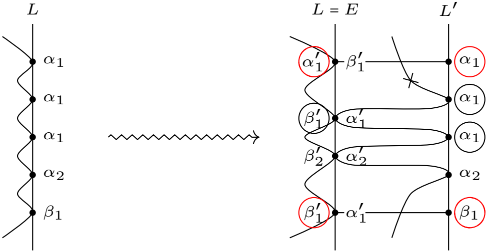
$d - d' = 1$ (GW Invariant in $\mathbb{F}$)



- GW invariant of the degree one curve is 1.
- GW invariant of each map of class mF is $1/m$.
  - Each element of α' is on a fiber.
  - Each element of β is on a fiber.

$$\Longrightarrow N_{\mathbb{F}}^{1,0}(\beta', \alpha'; \alpha, \beta) = \frac{1}{I^{\alpha'} I^{\beta}}$$

$d - d' = 1$ (Number of Graphs)



- On L' : each β is automatically on a fiber.
  - Each α' on L needs to be on a fiber.
  - $\binom{\alpha}{\alpha'}$ choices for which fixed points on L' lie on a fiber.
- On L : each α' is automatically on a fiber.
  - Each β on L' needs to be on a fiber.
  - $\binom{\beta'}{\beta}$ choices for which moving points on L lie on a fiber.

$d - d' = 1$ (Plugging In)

So in this case,

$$\sum_{\gamma/\sim} m(\gamma) \cdot N^{d',g'}(\alpha', \beta') \cdot N_{\mathbb{F}}^{d-d',g''}(\beta', \alpha'; \alpha, \beta)$$

$$= \sum \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} \cdot I^{\alpha'} I^{\beta'} \cdot N^{d-1,g'}(\alpha', \beta') \cdot N_{\mathbb{F}}^{1,0}(\beta', \alpha'; \alpha, \beta)$$

$$= \sum \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} \cdot I^{\alpha'} I^{\beta'} \cdot N^{d-1,g'}(\alpha', \beta') \cdot \frac{1}{I^{\alpha'} I^{\beta}}$$

$$= \sum \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} \cdot I^{\beta'-\beta} \cdot N^{d-1,g'}(\alpha', \beta').$$

This gives us the formula of Caporaso-Harris.

Recent Developments

1. Abramovich and Fantechi (2011) simplified the obstruction theory and extended the degeneration formula to orbifolds.
2. (a) Logarithmic GW invariants allow $D \subset Y$ to be a normal crossings divisor, as opposed to a smooth divisor in Jun Li's theory.

   [Chen and Abramovich (2011); Gross and Siebert (2011)]

   (b) Abramovich, Chen, Gross, and Siebert (2021) introduced punctured logarithmic maps that allow marked points to have negative contact orders.
3. There are degeneration techniques for rank 1 Donaldson-Thomas theory given by Li and Wu (2011) for smooth divisors and by Maulik and Ranganathan (2020) for simple normal crossings divisors.